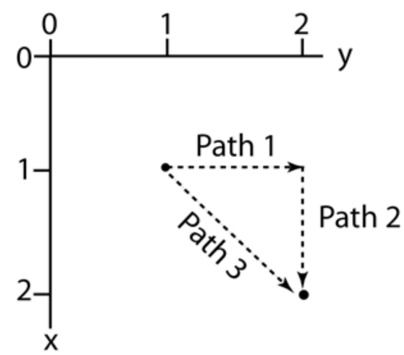
Exact vs Inexact Differnetial

For a function f(x,y) = x·y the partial of f with respect to x and y is

$$\partial f(x,y) = \frac{\partial f}{\partial x}\partial x + \frac{\partial f}{\partial y}\partial y$$

This is valid since I made no assumption whether x or y can change.
Since taking the derivative of f(x,y) is easy I can insert into the above:



$$\partial f(x,y) = y \cdot \partial x + x \cdot \partial y$$

Now for the change of f(x,y) from (x,y)=1,1 to (x,y)= 2,2 as per this diagram:

Path 1 + Path 2

Lets do Path 1 + Path 2 and compare to Path 3. Since I have partials I need to integrate:

$$\Delta f_1(x,y) = \int_{x=1}^{x=1} y \cdot \partial x + \int_{y=1}^{y=2} x \cdot \partial y$$

The first term is y·(1-1)=0 and the second is (since x=1) 1·(2-1)=1.
Now for path two

$$\Delta f_2(x,y) = \int_{x=1}^{x=2} y \cdot \partial x + \int_{y=2}^{y=2} x \cdot \partial y$$

Since y=2 the first term is 2·(2-1)=2 and the second is x·(2-2)=0. So the total is Δf_{1+2}=3.

Path 3

Lets do Path 1 + Path 2 and compare to Path 3. Since I have partials I need to integrate:

$$\Delta f_3(x,y) = \int_{x=y=1}^{x=y=2} y \cdot \partial x + \int_{x=y=1}^{x=y=2} x \cdot \partial y$$

The trick here is to recognize that since x = y you can replace every y with x and reduce this to:

$$\Delta f_3(x,y) = \int_{x=1}^{x=2} x \cdot \partial x + \int_{x=1}^{x=2} x \cdot \partial x = \int_{x=1}^{x=2} 2 \cdot x \cdot \partial x = x^2\Big]_1^2 = (4-1) = 3$$

In both cases Δf=3 so $\partial f(x,y) = y \cdot \partial x + x \cdot \partial y$ is an exact differential. This is a given since I had a functional form for f (f(x,y)=x·y) in the first place.

Inexact Differentials

In the case of $\partial f(x,y) = y \cdot \partial x$, which as I said in class is like force times distance, we can work the change in f as before:

Path 1 + Path 2

Lets do Path 1 + Path 2 and compare to Path 3. Since I have partials I need to integrate:

$$\Delta f_1(x,y) = \int_{x=1}^{x=1} y\,\partial x + \int_{y=1}^{y=2} 0\,\partial y$$

The first term is y·(1-1)=0 and the second is 0·(2-1)=0.
Now for path two

$$\Delta f_2(x,y) = \int_{x=1}^{x=2} y\,\partial x + \int_{y=2}^{y=2} 0 \cdot \partial y$$

Since y=2 the first term is 2· (2-1)=2 and the second is 0·(2-2)=0. So the total is Δf_{1+2}=2+0=2.

Path 3

Lets do Path 1 + Path 2 and compare to Path 3. Since I have partials I need to integrate:

$$\Delta f_3(x,y) = \int_{x=y=1}^{x=y=2} y\,\partial x + \int_{x=y=1}^{x=y=2} 0\,\partial y$$

Again using the same x = y trick:

$$\Delta f_3(x,y) = \int_{x=1}^{x=2} x\,\partial x + \int_{x=1}^{x=2} 0\,\partial x = \int_{x=1}^{x=2} x\,\partial x = \frac{1}{2}x^2\Big]_1^2 = \frac{1}{2}(4-1) = \frac{3}{2}$$

Hence $\partial f(x,y) = y \cdot \partial x$ is an inexact differential, and you should ***not*** be able to find a functional form for f(x,y) for which the partial derivative is y·δx. Thus the change in f(x,y) does depend on the path taken which is why work (w=-F·δx) and thermal energy (q=n·Cv·δT) are also inexact quantities.

Euler Criteria

Euler (pronounced oiler!) stated that for a differential $\partial f(x,y) = \frac{\partial f}{\partial x}\partial x + \frac{\partial f}{\partial y}\partial y$ to be exact then:

$$\frac{\partial}{\partial y}\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y}$$

This just says when taking the derivative of an exact function with respect to the variables, it doesn't matter which order you do the operations. In the case of f(P,V)=P·V, we get:

$$\frac{\partial}{\partial V}\frac{\partial (P \cdot V)}{\partial P} = \frac{\partial}{\partial P}\frac{\partial (P \cdot V)}{\partial V}$$

$$\frac{\partial}{\partial V}V = \frac{\partial}{\partial P}P$$

$$1 = 1$$

So yes P·V is an exact function.